CCFU Proof 9

Cassini Identity and H^1 Orbit

Given. $C_2 : x_{n+1} = x_n + x_{n-1}$.
Define (state-pair convention): $Q(x, y) = y^2 - xy - x^2$.

Claim. $Q(x_n, x_{n+1}) = (-1)^n \cdot Q(x_0, x_1)$ for all $n \geq 0$.

Proof by induction.
Base case ($n = 0$): $Q(x_0, x_1) = Q(x_0, x_1)$. ✓
Inductive step. Assume $Q(x_n, x_{n+1}) = (-1)^n \cdot Q(x_0, x_1)$. Then:

$$\begin{aligned}
Q(x_{n+1}, x_{n+2}) &= x_{n+2}^2 - x_{n+1}\, x_{n+2} - x_{n+1}^2 \\
&= (x_{n+1} + x_n)^2 - x_{n+1}(x_{n+1} + x_n) - x_{n+1}^2 \\
&= x_{n+1}^2 + 2x_{n+1}x_n + x_n^2 - x_{n+1}^2 - x_{n+1}x_n - x_{n+1}^2 \\
&= -x_{n+1}^2 + x_{n+1}x_n + x_n^2 \\
&= -(x_{n+1}^2 - x_{n+1}x_n - x_n^2) \\
&= -Q(x_n, x_{n+1}) \\
&= (-1)^{n+1} \cdot Q(x_0, x_1). \qquad \blacksquare
\end{aligned}$$

Corollary (H^1 orbit). For Fibonacci ($x_0 = 0$, $x_1 = 1$): $Q(0, 1) = 1$, so $Q(x_n, x_{n+1}) = (-1)^n$.
For general initial conditions:

$$Q(x_n, x_{n+1}) = (-1)^n\, Q(x_0, x_1).$$

Under the change of variables:

$$u = y - \tfrac{x}{2}, \qquad v = \tfrac{\sqrt{5}}{2}\, x.$$

Then:

$$u^2 - v^2 = (y - \tfrac{x}{2})^2 - \tfrac{5}{4}x^2 = y^2 - xy + \tfrac{x^2}{4} - \tfrac{5x^2}{4} = y^2 - xy - x^2 = Q(x, y).$$

Therefore all C_2 states lie on $u^2 - v^2 = (-1)^n\, Q(x_0, x_1)$ in Minkowski$(1, 1)$. $\blacksquare$